

TURNING BFKL

INSIDE OUT

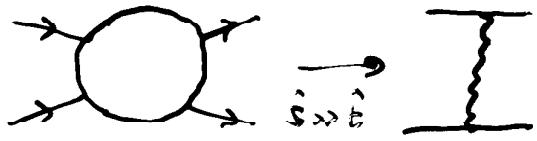
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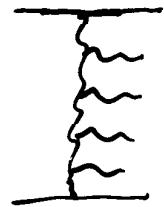
# What is BFKL?

(Balitsky-Fadin-Kuraev-Lipatov)

- ⊗ In scattering processes with  $\hat{s} \gg |\hat{t}|$ , gluon exchange in the  $\hat{t}$  channel dominates



- ⊗ BFKL is a resummation of multiple gluon radiation out of the gluon exchanged in the  $\hat{t}$  channel.



- \* Namely, for  $\hat{s} \gg |\hat{t}|$ , BFKL resums the Leading Log contributions, in  $\ln \frac{\hat{s}}{|\hat{t}|}$ , of the radiative corrections to the gluon propagator in the  $\hat{t}$  channel, to all orders in  $\alpha_s$ .
- \* The LL terms are obtained in the approximation of strong rapidity ( $\omega x$ ) ordering, and no  $K_1$  ordering, of the emitted gluons.
- \* The LL terms are universal
- \* The resummation yields a 2-dim. integral equation (BFKL eq.) for the evolution of the gluon propagator in the  $\hat{t}$  channel.

- ④ The BFKL eq. is scale invariant
  - ⊗ conformal invariance
  - ⊗ 2-dim. effective action
  - ⊗ exactly solvable models
- ⊗ Is BFKL realistic enough to describe QCD processes for  $|t| \gg 1$  ??
- ⊗ What about the NLL corrections ?

## Problems with BFKL

- ⊗ no energy conservation
- ⊗ no running of  $\alpha_s$
- ⊗ jets : no structure

$\Rightarrow$  quite primitive phenomenology

REM it's only a Leading Log resummation

improve ? compute Next-to-Leading Log corrections

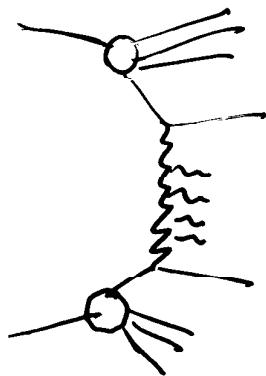
- ⊗ running  $\alpha_s$
- ⊗ jet structure
- ⊗ reduce energy-conservation violations

## Eventual applications



Growth of  $F_2$  in DIS  
at small  $x_b$

$$\ln \frac{S_{\text{rep}}}{Q^2} = \ln \frac{1}{x}$$



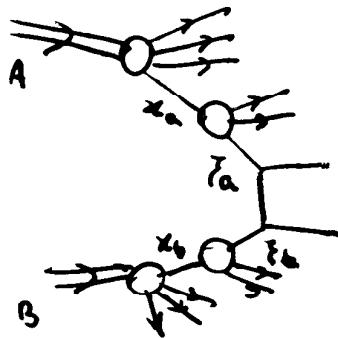
2-jet production in  $p - \bar{p}$  at  
large rapidity intervals  $\Delta y \approx \ln$   
(Mueller - Navelet jets)



Forward jet production  
in DIS

$$\ln \frac{S_{\text{rep}}}{Q^2} = \ln \frac{1}{z}$$

## Hard process



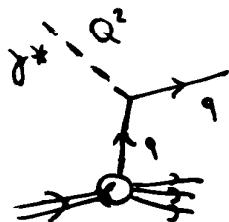
$$\sigma = \sum_{ab} \int dx_a \int dx_b f_{a/A}(x_a, \mu^2) f_{b/B}(x_b, \mu^2) \cdot \hat{G}_{ab}\left(\frac{x_a}{x_b}, \frac{\lambda_b}{\lambda_a}, \alpha_s(\mu^2), Q^2/\mu^2\right)$$

$\mu$  is factor

DGLAP evolution

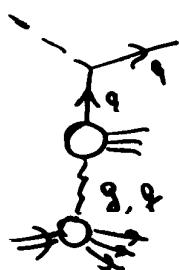
$$\frac{d\sigma}{d \ln \mu^2} = 0 \Rightarrow \frac{\partial f_{a/A}(x, \mu^2)}{\partial \ln \mu^2} = \sum_c \int_x^1 \frac{dy}{y} P_{ac}(y, \alpha_s) f_{c/A}\left(\frac{x}{y}, \mu\right)$$

DGLAP splitting functions



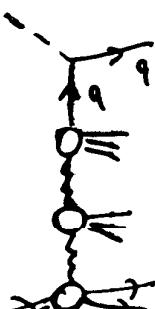
DIS (in DIS scheme)

$$F_2(x, Q^2) \sim x f_q(x, Q^2) + O\left(\frac{Q^2}{Q^2}\right)$$



$$\frac{\partial F_2}{\partial \ln Q^2} \sim P_{qg}(x, \alpha_s) \otimes f_g(x, Q^2) + P_{gg}(x, \alpha_s) \otimes f_g(x, Q^2)$$

mixing in x



$$\frac{\partial f_g(x, Q^2)}{\partial \ln Q^2} \sim P_{gg}(x, \alpha_s) \otimes f_g(x, Q^2) + P_{gq}(x, \alpha_s) \otimes f_q(x, Q^2)$$

$$P_{ac}(y, \alpha_s) = \sum_{n=1}^{\infty} \alpha_s^n P_{ac}^{(n)}(y)$$

$$\text{Diagram A} = \text{Diagram B} + \text{Diagram C} + \text{Diagram D} + \text{Diagram E}$$

$$P_{gg}(x, \alpha_s) = \alpha_s P^{(0)}(x) + \alpha_s^2 P^{(1)}(x) + \alpha_s^3 P^{(2)}(x) + \dots$$

DEFLAP

$$\beta_0 = \frac{11}{3} C_A - \frac{4}{3} m_F T_R$$

$$P_{gg}^{(1)}(z) = \frac{C_A}{\pi} \left[ \left( \frac{1}{1-z} \right)_+ - 1 + \frac{1-z}{z} + z(1-z) \right] + \frac{\rho_0}{4\pi} \delta(1-z)$$

$$\text{well } \Rightarrow P_{gg}^{(1)}(z) = \frac{c_1}{\pi} \frac{1}{z}$$

$$\alpha_s^n P_{gg}^{(n)}(z) \sim \frac{\alpha_s^n}{z} \left(\ln \frac{1}{z}\right)^{n-1} \quad \text{important if } \frac{1}{z} \gg 1$$

$$x^{fg} \sim \sum_{n=1}^{\infty} \frac{1}{n!} \left( \frac{c_n \alpha_s}{\pi} 4 \ln 2 \ln \frac{1}{x} \right)^n \sim x^{-4 \ln 2 \frac{c_n \alpha_s}{\pi}}$$

$$P_{ab}(z, \alpha_s) = \alpha_s P_{ab}^{(1)}(z) + \alpha_s^2 P_{ab}^{(2)}(z) + \sum_{n=3}^{\infty} A_{ab}^{(n)} \frac{\alpha_s}{z} \left( x_s \ln \frac{1}{z} \right)^{n-1} + \sum_{n=2}^{\infty} B_{ab}^{(n)} \frac{\alpha_s^2}{z} \left( x_s \ln \frac{1}{z} \right)^n$$

leaching for

West - 50 - Packard

$P_{gg}(z, \epsilon_s)$  Known at west-to-leading-log

Castanea - Hartigia

$$D(f_2 \circ f_1) \circ (f_2 \circ f_1) = 1 \in \mathbb{F}_q$$

$$\otimes \quad f_a(N, \mu^2) = \int_0^1 dx \ x^{N-1} f_a(x, \mu^2)$$

$$\otimes \quad P_{ab}(N+1, \alpha_s) = \int_0^1 dx \ x^N P_{ab}(x, \alpha_s) \equiv \gamma_{ab}(N, \alpha_s) \quad \text{anomalous dim.}$$

$$\otimes \quad \text{mtm. density} \quad \tilde{f}_a(x, \mu^2) = x f_a(x, \mu^2)$$

$$\otimes \quad \text{DGLAP eq.} \quad \frac{\partial \tilde{f}_a(N, \mu^2)}{\partial \ln \mu^2} = \sum_b \gamma_{ab}(N, \alpha_s) \tilde{f}_b(N, \mu^2)$$

small  $x$

$$P_{gg}^{(n)}(x) \sim \frac{1}{x} \left( \ln \frac{1}{x} \right)^{n-1} \Rightarrow \gamma_{gg}(N) \sim \int_0^1 dx \ x^{N-1} \left( \ln \frac{1}{x} \right)^{n-1} = \frac{1}{N^n}$$

anomalous dim.

$$\gamma_{ab}(N, \alpha_s) = \gamma_{ab}^{(1)}(N) \alpha_s + \gamma_{ab}^{(2)}(N) \alpha_s^2 + \sum_{n=3}^{\infty} A_{ab}^{(n)} \left( \frac{\alpha_s}{N} \right)^n + \sum_{n=2}^{\infty} B_{ab}^{(n)} \alpha_s \left( \frac{\alpha_s}{N} \right)^n$$

LL<sub>x</sub> (BFKL)

LL<sub>x</sub>

NLL<sub>x</sub>

gluon anomalous dim.

$$\gamma_{gg}(N, \alpha_s) = \gamma_{LL}(N, \alpha_s) + O\left(\alpha_s \left(\frac{\alpha_s}{N}\right)^n\right)$$

$$\text{defined by implicit eq.} \quad \frac{\bar{\alpha}_s}{N} \chi(\gamma_{LL}(\alpha_s)) = 1 \quad \bar{\alpha}_s = \frac{N \alpha_s}{\pi}$$

$$\text{w/} \quad \chi(y) = 2\psi(1) - \psi(y) - \psi(1-y)$$

$$\text{power series} \quad \gamma_{LL}(N, \alpha_s) = \frac{\bar{\alpha}_s}{N} + 2\gamma(3) \left( \frac{\bar{\alpha}_s}{N} \right)^4 + 2\gamma(5) \left( \frac{\bar{\alpha}_s}{N} \right)^6 + \dots$$

$$\gamma_{gg}(N, \alpha_s) = \frac{C_F}{C_A} \gamma_{q\bar{q}}(N, \alpha_s)$$

$\gamma_{q\bar{q}}$  &  $\gamma_{g\bar{q}}$  have no entry at  $LL_x$

LL<sub>x</sub>

② scheme dependence

generic scheme ( $\overline{MS}$  or else)

$$F_2(x, Q^2) = \sum_a \int_x^1 \frac{d\mu}{\mu} C_{2a}\left(\frac{x}{\mu}, \frac{Q^2}{\mu^2}, \alpha_s(\mu)\right) \tilde{f}_a(\mu)$$

parton model  $C_{2a}\left(\frac{x}{\mu}\right) = e_q^2 \delta\left(1 - \frac{x}{\mu}\right) + O(\alpha_s) = e_q^2 \gamma \delta(x - \mu) + O(\alpha_s)$

$$\Rightarrow F_2(x, Q^2) = \sum_q e_q^2 \tilde{f}_q(x) + O(\alpha_s)$$

DIS scheme

$$F_2(x, Q^2) = \sum_q e_q^2 \tilde{f}_q^{(DIS)}(x, Q^2) \rightarrow \text{all orders in } \alpha_s$$

ambiguity in DIS gluon density  $\Rightarrow$  (Catani-Hautmann) choose

$$\gamma_{ga}^{(DIS)}(N, \alpha_s) = \gamma_{ga}(N, \alpha_s) + O\left(\alpha_s^2 \left(\frac{\alpha_s}{N}\right)^n\right) \quad n = q, g$$

however  $\gamma_{qg}^{(DIS)}(N, \alpha_s) \neq \gamma_{qg}^{\overline{MS}}(N, \alpha_s) \rightarrow O\left(\alpha_s \left(\frac{\alpha_s}{N}\right)^n\right)$

scheme-independent color-charge relation

$$\gamma_{qg}^s(N, \alpha_s) = \frac{C_F}{C_A} \left( \gamma_{qg}^s(N, \alpha_s) - \frac{\alpha_s}{2\pi} \frac{2}{3} T_R \right) + O\left(\alpha_s^2 \left(\frac{\alpha_s}{N}\right)^n\right)$$

... if you are really curious...

in DIS scheme

Catani - Hautmann

$$\Upsilon_{qg}^{\text{DIS}}(N, \alpha_s) = \frac{\alpha_s}{2\pi} T_a \frac{2 + 3\gamma_{LL} - 3\chi_{LL}^2}{3 - 2\gamma_{LL}} \frac{\Gamma^3(1 - \gamma_{LL})}{\Gamma(2 + 2\gamma_{LL})} \frac{\Gamma^3(1 + \gamma_{LL})}{\Gamma(2 - 2\gamma_{LL})} R_\nu(\alpha_s)$$

$$+ O\left(\alpha_s^2 \left(\frac{\alpha_s}{N}\right)^6\right)$$

$$R_\nu(\alpha_s) = \left\{ \frac{\Gamma(1 - \gamma_{LL}) \chi(\gamma_{LL})}{\Gamma(1 + \gamma_{LL}) [-\gamma_{LL} \chi'(\gamma_{LL})]} \right\}^{1/2} \exp \left\{ \gamma_{LL} \psi(1) + \int_0^{\gamma_{LL}} dy \frac{\psi'(1) - \psi'(1-y)}{\chi(y)} \right\}$$

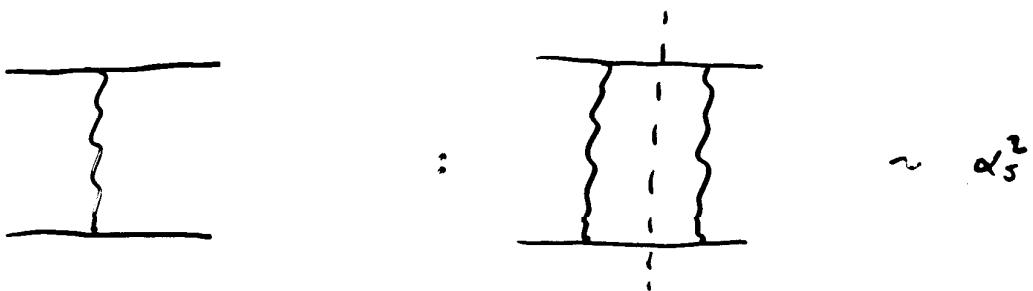
for  $\frac{\bar{\alpha}_s}{N} \ll 1$

$$\Upsilon_{qg}^{\text{DIS}}(N, \alpha_s) = \frac{\alpha_s}{2\pi} \frac{2}{3} T_a \left\{ 1 + \frac{13}{6} \frac{\bar{\alpha}_s}{N} + \left( \frac{71}{18} - \zeta(2) \right) \left( \frac{\bar{\alpha}_s}{N} \right)^2 + \dots \right\}$$

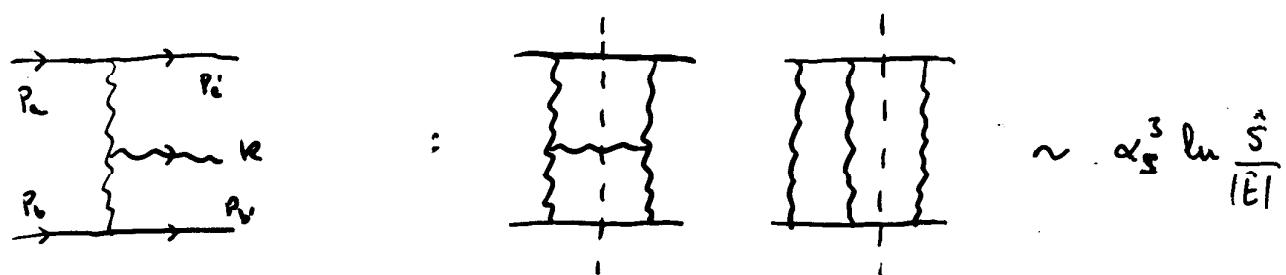
agree w/ Frixione - Peirano,

NLO

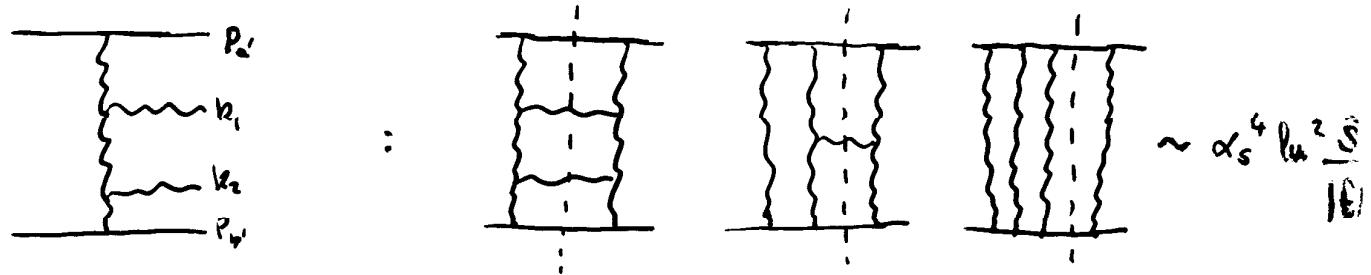
$\hat{S} \gg \hat{E}$ :



$Y_\alpha \gg Y \gg Y_\beta$ :



$\alpha_s(E) \gg Y_2 \gg Y_1$ :

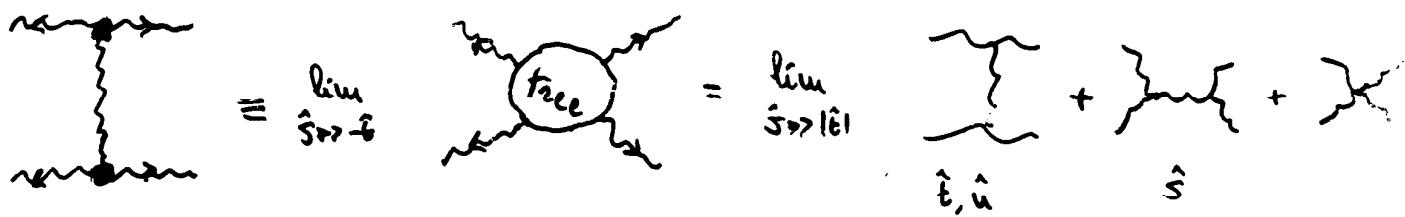


REH: No virtual corrections like

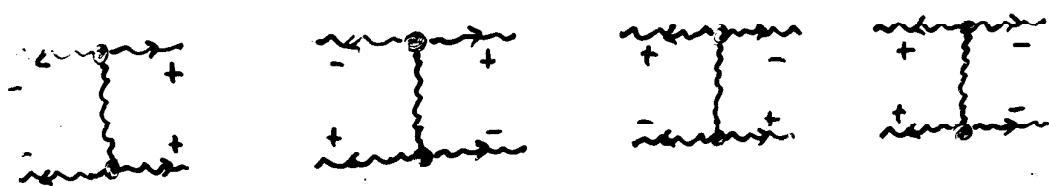


at LL  $\Rightarrow$  no contributions

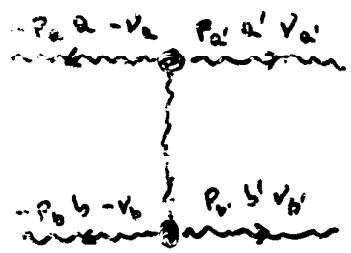
$\gamma\gamma \rightarrow \gamma\gamma$  scattering



leading helicity configurations



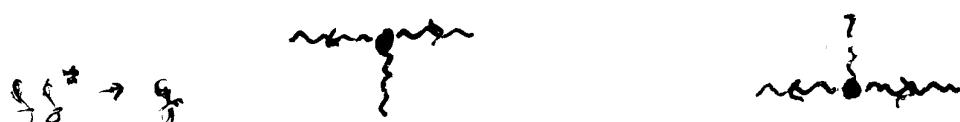
velocity conservation in the jet-production vertices



use PT amplitudes, take spinor products  
in the high-energy limit, recast traces of  $\lambda$ 's  
in terms of  $f^{abc}$ 's

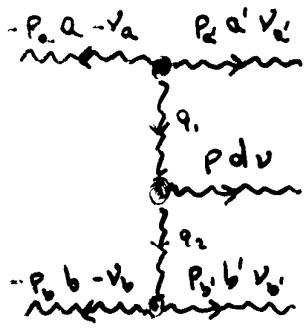
$$\langle \gamma \gamma | \bar{v}_a v_a \bar{v}_b v_b | \rangle = 2\hat{s} [ig f^{aa'c} C_{-+}^{gg}(-P_a, P_{a'})] \frac{1}{\epsilon} [ig f^{bb'c} C_{-+}^{gg}(-P_b, P_{b'})]$$

$$\text{w/ } C_{-+}^{gg}(-P_a, P_{a'}) = -1 \quad C_{-+}^{gg}(-P_b, P_{b'}) = -\frac{P_{a'}^+}{P_{b'}^+}$$



$$C_{+-}^{gg*} = C_{-+}^{gg} \quad \text{velocity flip} = \text{complex conjugation}$$

$g g \rightarrow g g g$  amplitude.



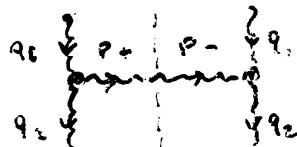
$\gamma_a \gg \gamma \gg \gamma_b$  multi Regge kinematics

$$M = 2 \hat{s} \left[ i g f^{a a' c} C_{-q_1 q_2}^{gg} (-p_a, p_{a'}) \right] \frac{1}{\hat{t}_1} \\ \times \left[ i g f^{c d e'} C_{+q_1}^g (q_1, q_2) \right] \frac{1}{\hat{t}_2} \\ \times \left[ i g f^{b b' c'} C_{-q_2 q_1}^{gg} (-p_b, p_{b'}) \right]$$

Lipatov vertex  $C_{+}^g (q_1, q_2) = \sqrt{2} \frac{q_1^\mu q_2_\mu}{k_1}$

sum over helicities

$$\sum_{s= \pm} C_{+}^g (q_1, q_2) C_{+}^{g*} (q_1, q_2) = 4 \frac{|q_{1s}|^2 |q_{2s}|^2}{|p_s|^2}$$



Linear vertex

$$C_\mu(q_i, q_{i+1}) = (q_i + q_{i+1})_i - P_b^\mu \left( \frac{\hat{S}_{ai}}{\hat{s}} + 2 \frac{\hat{t}_{i+1}}{\hat{s}_{ai}} \right) + P_a^\mu \left( \frac{\hat{S}_{bi}}{\hat{s}} + 2 \frac{\hat{t}_i}{\hat{s}_{bi}} \right)$$

$$\hat{t}_i = q_i^2 \approx -q_{i+1}^2$$

at fixed helicity  $C(q_i, q_{i+1}) \cdot \epsilon^+(p_i, p_a) = \sqrt{2} \frac{q_i^+ q_{i+1}^-}{p_i^+}$

$$\sum_{\nu=\pm} \epsilon^+(p_i) \cdot C(q_i, q_{i+1}) \epsilon^{*\nu}(p_i) \cdot C(q_i, q_{i+1}) = 4 \frac{|q_i|^2 |q_{i+1}|^2}{|p_i|^2}$$

Helicity-conserving vertex

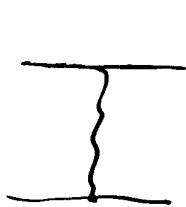
$$C^{M\mu\nu\rho}(p_a, p_b, p_c) = g^{M\mu\nu\rho} - \frac{p_a^M p_b^{\mu\nu\rho}}{p_a \cdot p_b} - \frac{p_b^M p_c^{\mu\nu\rho}}{p_b \cdot p_c} + p_c^M p_a^{\mu\nu\rho} \frac{p_a \cdot p_b}{p_a \cdot p_b p_b \cdot p_c}$$

at fixed helicity  $C^{M\mu\nu\rho}(p_a, p_b, p_c) \epsilon_{\mu\nu\rho}^+(p_a, p_b) \epsilon_{\mu\nu\rho}^-(p_c, p_b) = -1$

FCC amplitudes at fixed helicities are very simple

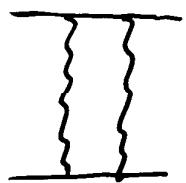
$$M = 2 \hat{s} \left[ i g f^{a\bar{c}, c\bar{d}} C_{\nu_1 \nu_2}^{gg}(-p_a, p_{a\bar{c}}) \right] \frac{1}{\hat{E}_1} \\ \times \left[ i g f^{c\bar{d}, d\bar{a}} C_{\nu_1}^g(q_1, q_2) \right] \frac{1}{\hat{E}_2} \\ \vdots \\ \times \left[ i g f^{c\bar{d}, d\bar{a}} C_{\nu_m}^g(q_m, q_{m+1}) \right] \frac{1}{\hat{E}_{m+1}} \\ \times \left[ i g f^{b\bar{b}', b'\bar{c}} C_{-\nu_1 \nu_2}^{gg}(-p_b, p_{b\bar{b}'}) \right]$$

# VIRTUAL RADIATIVE CORRECTIONS



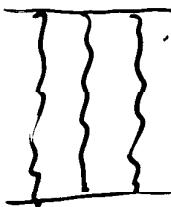
$$\hat{s} \gg |\hat{E}|$$

$$M \sim \frac{\hat{s}}{|\hat{E}|}$$



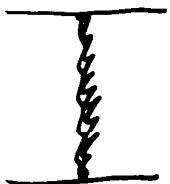
$$M \sim \frac{\hat{s}}{|\hat{E}|} \ln \frac{\hat{s}}{|\hat{E}|} \alpha(t)$$

$$\text{w/ } \alpha(t) = \alpha_s N_c \hat{t} \int \frac{d^4 k}{(2\pi)^4} \frac{1}{k_\perp^2 (q - k)_\perp^2}; \quad q_\perp^2 \approx -\hat{t}$$



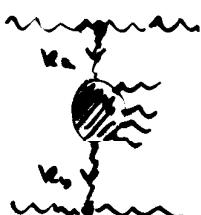
$$M \sim \frac{\hat{s}}{|\hat{E}|} \left[ \frac{1}{2} \ln^2 \frac{\hat{s}}{|\hat{E}|} \alpha^2(t) \right]$$

all orders in  $\alpha_s$



$$M \sim \frac{\hat{s}}{|\hat{E}|} e^{\alpha(t) \ln \frac{\hat{s}}{|\hat{E}|}}$$

The BFKL eq. describes the evolution of a gluon Green's fn. in  $k_\perp$  & momenta space in the  $t$  channel



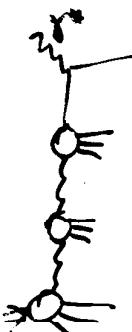
$$\omega_{fw}(k_a, k_b) = \frac{1}{2} \delta^2(k_a - k_b) + \frac{\bar{\alpha}_s}{\pi} \int d^2k \frac{1}{(k_a - k)^2} K(k, k_a, k_b)$$

$$\text{w/ kernel } K(k, k_a, k_b) = f_w(k, k_b) - \frac{k_a^2}{[k^2 + (k_a - k)^2]} f_w(k_a, k_b)$$

$$\text{definition } f_{\omega}(k_a, k_b) = \frac{1}{(2\pi)^2} \frac{1}{k_a k_b} \sum_{n=-\infty}^{\infty} e^{in\varphi} \int_{-\infty}^{\infty} dv \frac{e^{iv \ln k_a^2/k_b^2}}{\omega - \omega(n, v)}$$

$$w/ \quad \omega(n, v) = 2 \bar{\alpha}_s \left[ \psi(1) - \operatorname{Re} \psi \left( \frac{k_m + 1}{2} + iv \right) \right]$$

D15



$$\frac{\partial \tilde{f}_g(\omega, \mu^2)}{\partial \ln \mu^2} = \gamma_{gg}(\omega, \alpha_s) \tilde{f}_g(\omega, \mu^2) + \gamma_{gq}(\omega, \alpha_s) \tilde{f}_q(\omega, \mu^2)$$

\* to get LL<sub>c</sub> contribution to  $\gamma_{gg}$ , average SFKL solution

are  $\Psi$ , and fix  $\gamma = \frac{1}{2} + iv$

$$\Rightarrow \Phi_{\omega}(k_x, k_y) = \frac{1}{(2\pi)^2} \cdot \frac{1}{k_x^2} \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi i} \frac{1}{\omega - \omega(0, v_j)} \left(\frac{k_x^2}{k_0^2}\right)^j$$

$$\text{w/ } \omega(0, v) = \omega(\gamma) \equiv \bar{\alpha}_3 \chi(\gamma) \Rightarrow \chi(\gamma) = 2\psi(1) - \psi(\gamma) - \psi(1-\gamma)$$

$$x + \gamma = \bar{x}_1 \gamma'(x) + 0 \quad \Rightarrow \quad \bar{x}_1 \gamma'(x) = 1$$

\* Use BFKL solution, i.e. the  $k_t$ -dependent Green's fn.,  
as an unintegrated gluon density, then may write

$k_t$ -factorization (Cetani - Cieplinski - Hentschmann)

DIS

$$F_i(x, Q^2) = \int_x^1 \int d\gamma \int d^2 k_t \hat{C}_i^g\left(\frac{x}{\gamma}, \frac{k_t^2}{Q^2}, \alpha_s\right) \hat{f}_g(\gamma, k_t^2, \mu^2)$$

w/  $\int d^2 k_t \hat{f}_g(\gamma, k_t^2, \mu^2) = \int f_g(\gamma, \mu^2)$   $i = 2, L, \dots$

$\hat{C}_i^g$  describes  $\gamma^* g^* \rightarrow q\bar{q}$

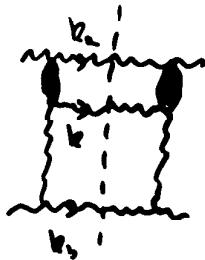


Callan - Gross ( $F_L(x) = 0$ )  $\Rightarrow \hat{C}_L^g$  collinear finite

but  $\hat{C}_2^g$  has collinear pole (small-x dominance appears at NLO)  
 $\Rightarrow$  need to embed  $k_t$ -factorization into usual collinear factorization  
(Cetani - Hentschmann)

\*  $\hat{f}_g(\gamma, k_t^2, \mu^2) = \int \frac{dw}{2\pi i} \gamma^{-w} f_w(k_t^2, \mu^2)$

NEXT - PC-LIA SIVRE



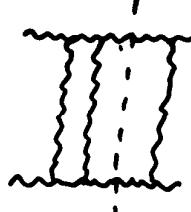
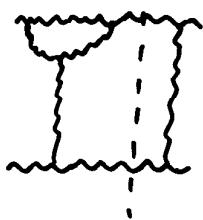
$$\eta_a \approx \eta \gg \eta_b$$

real corrections to helicity-cons. vertex

Fadin-Lipatov '89

VDD '96

$\mathcal{O}(\alpha_s^3)$

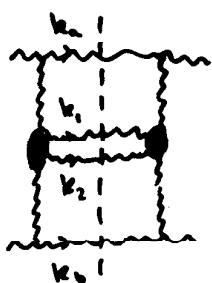


1-loop corrections to helicity-cons. vertex

Fadin-Fiori '92

Fadin-Lipatov '93

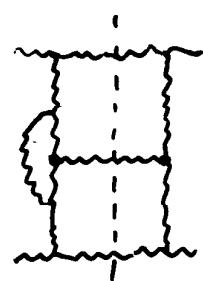
SEM: NL corrections introduce running of  $\alpha_s$



$$\eta_a \gg \eta_1 \approx \eta_2 \gg \eta_3$$

Fadin-Lipatov '89 / '96  
VDD '96

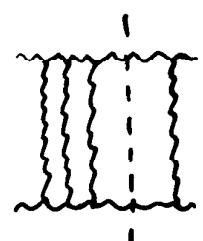
real corrections to Lipatov vertex



Fadin-Lipatov '93

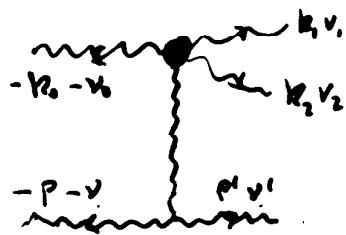
1-loop corrections to Lipatov vertex

$\mathcal{O}(\alpha_s^4 \ln \frac{s}{\mu^2})$



Fadin-Fiori-Duraturo '94-'95

2-loop corrections to regularization of gluon



$$\begin{aligned}
 & M^{\delta\delta}(-k_0 - v_0; k_1 v_1; k_2 v_2; p' v'; -p - v) = \\
 & = 2 \hat{s} \left\{ C_{-v_0 v_1 v_2}^{333}(-k_0, k_1, k_2) \left[ (ig)^2 f^{ddc} f^{cd_2 c'} \frac{1}{\sqrt{2}} A_{\Sigma v}(k_1, k_2) + \left( \begin{array}{l} k_1 \leftrightarrow k_2 \\ d_1 \leftrightarrow d_2 \end{array} \right) \right] \right\} \\
 & \cdot \frac{1}{\hat{t}} \left[ ig f^{ddc'} C_{-v v'}^{33}(-p, p') \right]
 \end{aligned}$$

$\sum v = -v_0 + v_1 + v_2$

related by a SUSY Ward identity to

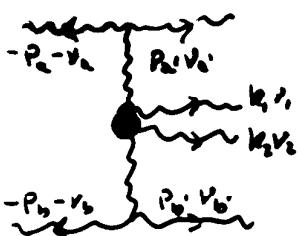
$$\begin{aligned}
 & \text{new } \begin{array}{c} \text{---} \\ \text{---} \end{array} \rightarrow \begin{array}{c} k_1 v_1 \\ k_2 v_2 \end{array} \quad k_1 = \text{antiquark} \\
 & \begin{array}{c} \text{---} \\ \text{---} \end{array} \rightarrow \begin{array}{c} -p - v \\ p' v' \end{array} \quad M^{\bar{q}q}(-k_0 - v_0; k_1 v_1; k_2 v_2; p' v'; -p - v)
 \end{aligned}$$

$$\begin{aligned}
 & = 2 \hat{s} \left\{ \sqrt{2} g^2 C_{-v_0 v_1 v_2}^{3\bar{q}q}(-k_0, k_1, k_2) \left[ (\lambda^c \lambda^{d0})_{;\bar{q}} A_{-v_0}(k_1, k_2) + (\lambda^{d0} \lambda^c)_{;\bar{q}} A_{-v_0}(k_2, k_1) \right] \right. \\
 & \left. \cdot \frac{1}{\hat{t}} \left[ ig f^{ddc'} C_{-v v'}^{33}(-p, p') \right] \right\}
 \end{aligned}$$

$$\text{w/ } A_+(\mathbf{k}_1, \mathbf{k}_2) = 2 \frac{p'_1}{k_{1z}} \frac{1}{k_{2z} - k_{1z} \frac{k_2^2}{k_1^2}}$$

COLLECTIONS IN THE CENTRAL REGION

Parker-Taylor '25  
Gersmehl-Gieke '27



Mangano-Parkle-Xu '88

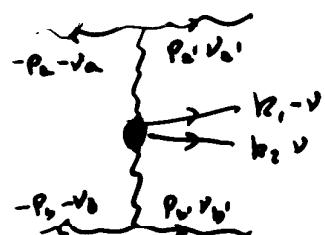
VDA '96

Fadin-Lipatov '83-'96

} high-energy limit

$$M^{gg} = 2\hat{S} \left[ ig f^{a c c'} C_{-\nu_a \nu_a'}^{gg}(-p_a, p_c) \right] \frac{1}{\hat{t}_a} \left[ (ig)^2 f^{c d, j} f^{j d, c'} A_{\nu, \nu_2}^{gg}(k_1, k_2) + \begin{pmatrix} \nu_1 \leftrightarrow \nu_2 \\ k_1 \leftrightarrow k_2 \\ d_1 \leftrightarrow d_2 \end{pmatrix} \right]$$

$$\times \frac{1}{\hat{t}_b} \left[ ig f^{b b' c'} C_{-\nu_b \nu_b'}^{gg}(-p_b, p_{b'}) \right]$$



$$M_{\bar{q}q} = 2\hat{S} \left[ ig f^{a c c'} C_{-\nu_a \nu_a'}^{gg}(-p_a, p_c) \right] \frac{1}{\hat{t}_a} g^2 \left[ (\lambda^c \lambda^c)_{ij} \bar{A}_{\nu\nu}^{q\bar{q}}(k_1, k_2) - (\lambda^c \lambda^{c'})_{ij} \bar{A}_{\nu\nu}^{q\bar{q}}(k_2) \right.$$

$$\left. \times \frac{1}{\hat{t}_b} \left[ ig f^{b b' c'} C_{-\nu_b \nu_b'}^{gg}(-p_b, p_{b'}) \right] \right]$$

## 1-Loop Helicity Amplitude

REGULARIZATION FACTOR

$$\alpha(\hat{t}) = \alpha_s N_c \hat{t} \int \frac{d^4 k}{(2\pi)^4} \frac{1}{k_i^2 (q-k)_i^2}$$

$$D = 4 - 2\epsilon$$

$$\alpha(t) = g^2 N_c \frac{2}{\epsilon} \left( \frac{-\hat{t}}{\mu^2} \right)^{-\epsilon} \frac{(4\pi)^\epsilon}{16\pi^2} \underbrace{\frac{\Gamma(1+\epsilon) \Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}}_{C_F}$$

## 4-Gluon 1-Loop Amplitude

BERN-KOSOWER NP B129 '92  
KUNZETZ-SIGNER-TROCSANYI NP B411 '93

$$M^{loop}(B^-, A^-, A'^+, B'^+) = M_{tree} g^4 C_F$$

$$\left\{ \left[ N_c \left( -\frac{4}{\epsilon^2} + \frac{2}{\epsilon} \ln \frac{S}{-t} - \frac{11}{3\epsilon} + \pi^2 - \frac{64}{9} - \frac{8\pi}{3} \right) + N_f \left( \frac{2}{3\epsilon} + \frac{10}{9} \right) \right] \right. \\ \left. - \frac{\beta_0}{\epsilon} \right\}$$

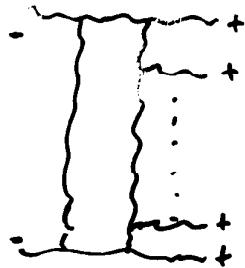
$\leftarrow \bar{MS}$  UV counterterm

$$\delta_\alpha = \begin{cases} 1 & \text{CDR / HV} \\ 0 & \text{dim. reduction} \end{cases}$$

agrees at LL & NLL accuracy with Fadin-Lipatov

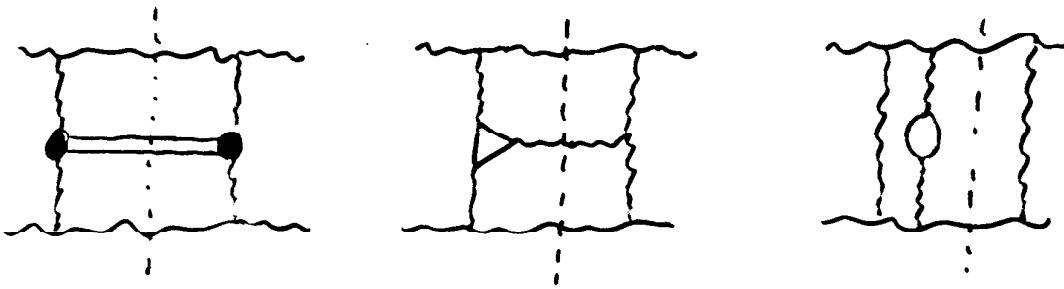
$N=4$  SUSY  
FT gluon 1-loop amplitude

Breidenbach-Kosower NP B425 ('95)



agree at  $LL_\alpha$  accuracy with  
FL amplitudes, because  
in the high-energy limit at  $LL_\alpha$  accuracy  
QCD &  $N=4$  SUSY QCD amplitudes  
coincide:  
only gluon exchange in the  $\hat{t}$  channel

# QUARK-LOOP CONTRIBUTION TO $\gamma_{gg}$



$N_f$ -dependent part of  $\gamma_{gg}$

Canetti - Ciafaloni

- \* assume Kernel of BFKL-type eq. to be

$$K_\omega(k, k') = \frac{\bar{\alpha}_s(k^2)}{\omega} \left( K_0(k, k') + \alpha_s K_1(k, k') \right) \quad \text{NLL}$$

w/  $\int \frac{d^2 k'}{\pi} (k')^{Y-1} K_\omega(k, k') = \chi(\gamma) (k^2)^{Y-1}$

whose saddle pt. are given by the solution of

$$*, \quad \frac{\bar{\alpha}_s(k^2)}{\omega} [x_0(\gamma) + \alpha_s x_1(\gamma)] = 1 \quad \bar{\alpha}(k^2) \equiv 1\text{-loop running}$$

- \* saddle pt's located at 2 eigenvalues of anomalous dim. matrix

$$\begin{vmatrix} \gamma_{99} - \lambda & \gamma_{9g} \\ \frac{C_F}{C_A} \gamma_{gg} & \gamma_{gg} - \lambda \end{vmatrix} = 0 \quad \Rightarrow \quad \begin{cases} \gamma_+ \simeq \gamma_{gg} + \frac{C_F}{C_A} \gamma_{9g} & \rightarrow \text{LL + NLL} \\ \gamma_- \simeq \gamma_{99} - \frac{C_F}{C_A} \gamma_{9g} \end{cases}$$

- \* introduce modified coupling  $\tilde{\alpha}_s = \bar{\alpha}_s \left( 1 + \frac{\chi_1(\frac{1}{2})}{\chi_0(\frac{1}{2})} \right)$  into eq. \*

$$\frac{\tilde{\alpha}_s}{\omega} \left\{ \chi_0(\gamma_+) + \alpha_s [\chi_1(\gamma_+) - \chi_1(\gamma_2)] \right\} = 1$$

\* expand  $\chi_0$  around  $\gamma_+ = \gamma_{LL}$ , get

$$\begin{cases} \frac{\tilde{\alpha}_s}{\omega} \chi_0(\gamma_{LL}) = 1 \\ \gamma_+ - \gamma_{LL} = \alpha_s \frac{\chi_1(\gamma_{LL}) - \chi_1(\gamma_2)}{-\chi_0'(\gamma_{LL})} \end{cases}$$

\* so compute  $\chi_1$ , and use  $\uparrow$  to derive  $\gamma_+ - \gamma_{LL}$

Caveat: calculations done in  $Q_0$  scheme, i.e. a modification of DIS scheme which keeps initial gluon virtuality fixed at  $k^2 = Q_0^2$

Ciafaloni '95

$$K_0 + \alpha_s K_1^{qq} = \left( 1 + \frac{\alpha_s N_F}{6\pi} \ln \frac{k^2}{\lambda^2} \right) K_0(k, k') + \frac{\alpha_s N_F}{6\pi} \left[ \left( \ln \frac{q^2}{k^2} - \frac{5}{3} \right) K_0(k, k') - \frac{1}{k^2 - k'^2} \ln \frac{k^2}{k'^2} \right] + K_{reg}^{qq}(k, k')$$

$$w, q = k - k'$$

contains di-logs

get

$$K_1^{qq} = \frac{\alpha_s N_F}{6\pi} \left[ \frac{1}{2} (\chi'_0(\gamma) + \chi''_0(\gamma)) - \frac{5}{3} \chi_0(\gamma) - \frac{1}{N_c^2} \left( \frac{\pi}{\sin \pi \gamma} \right)^2 \frac{3 \cos(\pi \gamma) [1 + \frac{3}{2} \gamma(1-\gamma)]^2}{(3-2\gamma)(1-2\gamma)(1+2\gamma)} \right]$$

expand, obtain

$$\gamma_{LL} = - \frac{\alpha_s N_F}{1 + \frac{23}{2} \tilde{\alpha}_s + \dots}$$

in agreement w/ NLO FeynmanZ - Petrouzos, ..

Scheme dependence

	Catani-Hautmann	Cacciari-Catani
UV subtraction	$\overline{MS}$	$\overline{MS}$
mass factorization	$\overline{MS}$	$Q_0$
pdf choice	$\Delta MS$ or $\overline{MS}$	$\Delta MS$

Cannice - Lieffler '87

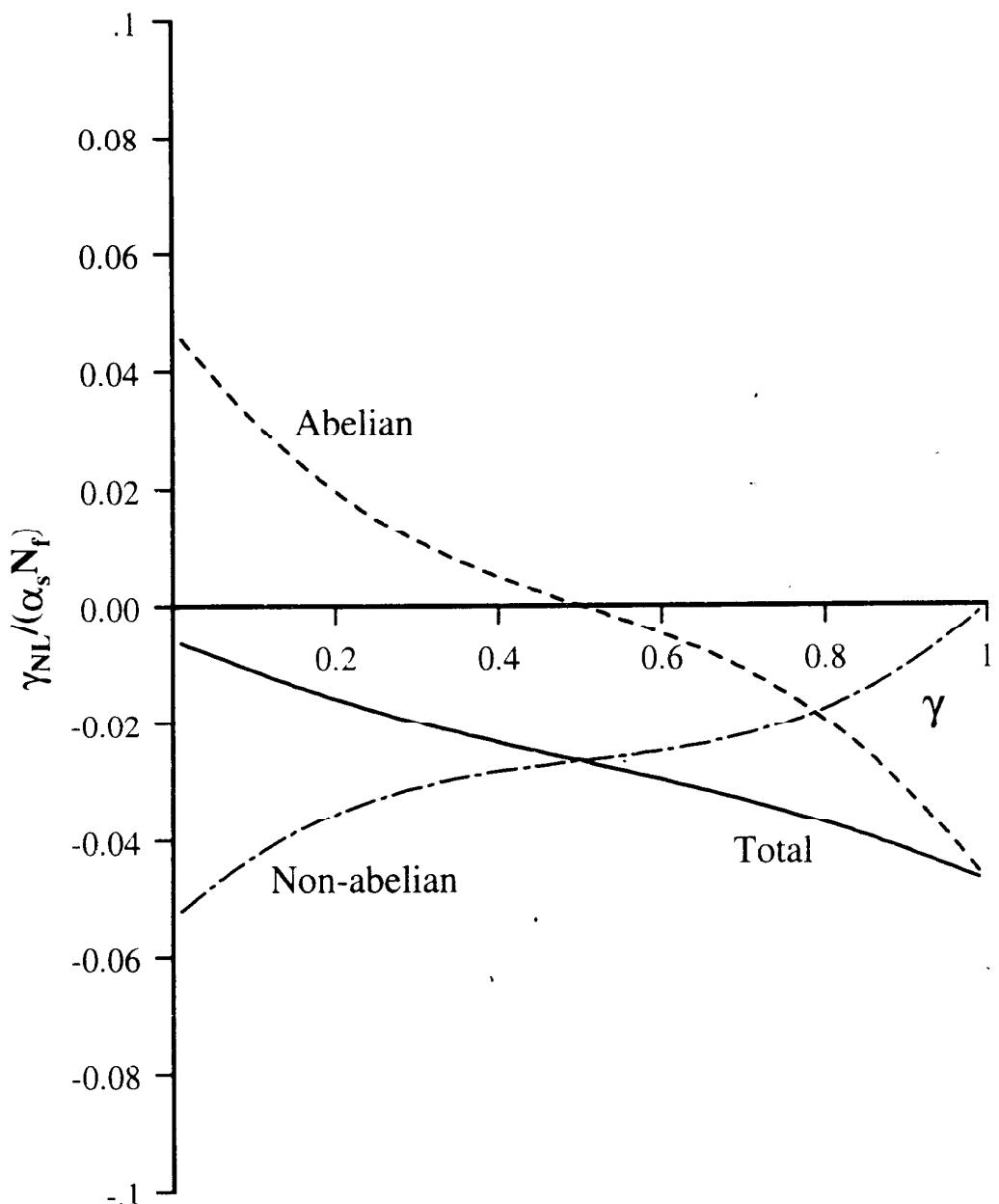


Figure 6:  $q\bar{q}$  contribution to the largest eigenvalue of the anomalous dimension matrix

## Conclusions

- ⊗ BFKL, i.e. LL<sub>x</sub> resummation:  
we know more and more about it
- ⊗ Other analyses which have BFKL as a spinoff
  - ⊗ Angular ordering at small  $x$  (CCFM)
  - ⊗ Dipole formalism in quark-quark scattering (A. Mueller)
  - ⊗ Renormalization-group evolution in  $x$  (Blaeck-Forte)
  - ⊗ Monte Carlo solution to BFKL (C. Schmidt)  
(it's got energy-mtm. conservation!)

## NLL<sub>x</sub> resummation

- ⊗  $\gamma_{gg}$ ,  $\gamma_{qg}$  are known (in  $\overline{\text{MS}}$  & DIS scheme)
- ⊗  $q\bar{q}$  part of  $\gamma_{gg}$ ,  $\gamma_{qg}$  is known (in QCD-DIS scheme)  
(for gg part, see next speaker)
- ⊗ possible to define physical, i.e. scheme-independent anomalous dim.  
(Catani '96)
- ⊗ desirable a simpler and neater derivation of NLL<sub>x</sub> terms